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Research articles

Elementary students' functional thinking in solving context-based linear pattern problems

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Abstrak Penelitian ini bertujuan untuk mengidentifikasi proses berpikir fungsional siswa Sekolah Dasar (SD) dalam menyelesaikan soal pola linier berbasis konteks. Partisipan penelitian terdiri dari 65 siswa kelas 5 yang belum memperoleh materi generalisasi pola. Data dikumpulkan melalui tes dan wawancara. Siswa dengan jawaban benar pada tes dikelompokkan menjadi tiga kategori berdasarkan tipe berpikir fungsional dan representasi yang digunakan. Wawancara dilakukan dengan tiga siswa terpilih untuk mengidentifikasi proses berpikir terkait aksi dan refleksi. Hasil penelitian menunjukkan bahwa siswa SD mampu berpikir fungsional yang terdiri dari tiga jenis, yaitu berpikir fungsional *recursive-verbal*, *correspondence-verbal*, dan *recursive to correspondence-symbolic*. Berpikir fungsional *recursive-verbal* dilakukan dengan menjumlahkan bilangan yang sama secara berulang dan merepresentasikan hasil generalisasi secara verbal. Berpikir fungsional *correspondence-verbal* dilakukan dengan menentukan hubungan dua kuantitas dengan pola tertentu dan direpresentasikan secara verbal. Sedangkan berpikir fungsional kategori *recursive to correspondence-symbolic* dilakukan dengan generalisasi secara rekursif kemudian dilanjutkan dengan proses generalisasi secara koresponden dan merepresentasikan hubungan tersebut secara simbolik. Ketiga jenis berpikir fungsional tersebut dilakukan dengan tahapan generalisasi aksi dan refleksi.

Kata kunci *Berpikir fungsional, Pola linier, Konteks, Aksi, Refleksi*

Abstract This study aims to identify elementary students' functional thinking in solving context-based linear pattern problems. It involved 65 fifth-grade students who had not learned generalizing patterns topic. Data was collected through tests and interviews. Students with correct answers on the test were grouped into three categories based on the types of functional thinking and the use of representations. Interviews were conducted with three selected students to identify their thinking relating to actions and reflections. The findings showed that elementary students are able to think functionally, consisting of three types: recursive-verbal, correspondence-verbal, and recursive to correspondence-symbolic. The students with recursive verbal add up the same numbers repeatedly and represent the generalization results verbally. For correspondence-verbal students, the relationship between two quantities with a certain pattern is determined and represented verbally. The students having recursive to correspondence-symbolic develop recursive generalization and then continue with the correspondence generalization and represent the relationship symbolically. The generalization of action and reflection is also identified in the students' functional thinking.

Keywords *Functional thinking, Linear pattern, Context, Action, Reflection*

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Introduction

Many students experience difficulties in algebra (Kieran, 2007). Wijaya et al. (2014) revealed that 14% of 367 students experienced errors in the mathematics process for the algebraic domain. Moreover, Stacey (2011) shows that algebra and calculation problems in PISA were more difficult for Indonesian students to solve than questions about numbers, geometry, and data. Subanji and Nusantara (2013) found that students experienced a pseudo error in solving algebra problems. The pseudo consists of pseudo true and pseudo false. The first occurs when students answer correctly but cannot give reasons for their answers, while the latter happens when students answer incorrectly, but after reflection, students are able to correct their answers. The results of the studies emphasize the importance of inquiring and improving students' abilities in algebra.

Algebra is one of the mathematical content standards (the Ministry of Educations and Culture (MoEC), 2016; NCTM, 2000) and it plays a paramount role in mathematics education (Kieran, 2007). The algebraic competencies including understand patterns, relationships, and functions. The competencies aim for students to have basic abilities in algebraic reasoning. It is a process of generalizing mathematical ideas from a set of specific examples, establishing these generalizations through a discourse of argumentation, and expressing them in an increasingly formal and age-appropriate manner (Blanton & Kaput, 2005).

Blanton and Kaput (2005) classify algebraic thinking into three categories: arithmetic generalizations, functional thinking, and generalization and justification. Functional thinking is a part of algebraic reasoning (Blanton & Kaput, 2005; Smith, 2008). It is fundamental for algebraic thinking (Eisenmann, 2009; Kaiser & Willander, 2005; Lichti & Roth, 2018, 2019; Stephens et al., 2017) because it involves the generalization of how quantities are related (Tanışlı, 2011) and a central topic in mathematics.

Students are able to think functionally at the elementary school level (Blanton & Kaput, 2004, 2005, 2011; Warren et al., 2006; Warren & Cooper, 2005). In kindergarten, students are able to think variationally. The students can think correspondently in the first grade and symbolically in the third grade (Blanton & Kaput, 2004). In the fourth grade, they are able to develop functional thinking verbally and symbolically (Warren et al., 2006).

Understanding patterns, relationships, and functions as a basis for the development of functional thinking are taught from grade 3 to grade 5 in elementary schools (NCTM, 2000). However, in the Indonesian curriculum (K-13) (MoEC, 2016), those competencies began to be taught in secondary schools. The mathematical curriculum for elementary schools in Indonesia emphasizes the strengthening of numbers, geometry, and data. Despite the difference, it is suspected that elementary students in Indonesia are able to think functionally when given a problem, as the previous studies (Blanton & Kaput, 2011; Warren et al., 2006) claimed that elementary students are able to think functionally. Therefore, the current study conjectured that elementary school students in Indonesia are also able to think functionally. Unlike previous studies, this study did identify not only the ability of elementary school students to think functionally but also describes the thinking processes carried out by the students in solving linear pattern-based context problems. The research questions sought to be answered: (1) are elementary students in Indonesia able to think functionally? and (2) how are the students' functional thinking processes?

Theories about Functional Thinking

Functional thinking is generalizing relationships between co-varying quantities, expressing those relationships in words, symbols, tables, or graphs, and reasoning with these various representations to analyze the function behavior (Blanton et al., 2011). Smith (2008) defines functional thinking as representational thinking that focuses on the relationship between two (or more) varying quantities, specifically the kinds of thinking that lead from specific relationships (individual incidences) to generalizations of that relationship across instances. According to Stephens et al. (2017), functional thinking is the process of building, describing, and reasoning with and about functions. Blanton (2008) asserts that functional thinking closely relates to the function concept.

Functional thinking consists of three types: recursive patterning, covariational thinking, and correspondent relationship (Confrey & Smith, 1991; Smith, 2008). Recursive patterning involves finding variation within a sequence of values. Covariational thinking is based on analyzing how two quantities vary simultaneously and keeping that change as an explicit, dynamic part of a function's description. The correspondence relationship is based on identifying a correlation between variables.

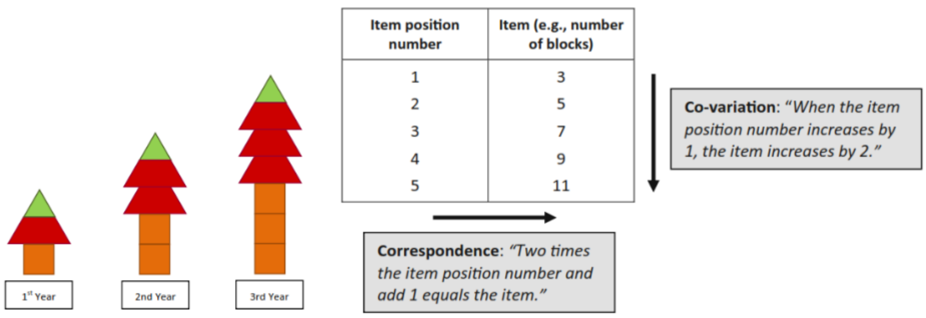


Figure 1. Examples of the types of functional thinking

Figure 1 represents an example of each type of functional thinking. The first-year pine tree has three blocks, and each tree has two more blocks than the previous one. The total number of blocks is 3, 5, 7, … and it has a pattern of increasing two more blocks every year (Recursive patterning). In covariational thinking, two items need attention, one item for year number and another item for blocks number. It is shown by the item year increases by 1, then the item of blocks number increase by 2. In a correspondent relationship, each pine tree has the same number of squares as its item number and the same number of trapeziums as its item number. It is represented symbolically by $t=2n+1$, where t is the item for the block number and n is the item for the year.

Assessing Functional Thinking

Functional thinking is a particular kind of generalizing thinking to develop algebraic thinking. It is a type of representational thinking that focuses on the relation between two quantities (Smith, 2008). According to NCTM (2000), grade 3-5 students should be able to (a) describe, extend, and make generalizations about geometric and numeric patterns and (b) represent and analyze patterns and functions using words, tables, and graphs. These are the basis for assessing students' functional thinking.

Several studies used various methods in assessing students' ability in functional thinking. Blanton and Kaput (2004) used a context-based relationship test in classroom observation. A context-based relationship test carries out a relation between a dog's number and the total number of eyes and tail. Warren et al. (2006) used a test about a picture of a function machine with tables IN and OUT. Tanıslı (2011) used the task-based interview to describe students' work with function tables and to investigate students' ways of functional thinking. One of the task-based interviews includes a task about the relation between the number of triangles and squares. Wilkie and Clarke (2016) applied a figural linear and non-linear task-based context. Stephens et al. (2017) provide problems in determining the relationship between days and the number of circles.

In this study, linear pattern-based context problems were used to examine elementary students' functional thinking along with follow-up interviews. The problems include a relationship between the number of days and people adopted from Wilkie and Clarke (2016). The reason for selecting the problems is that it includes a figural linear pattern-based context which is easy for elementary students.

The process of students' functional thinking was further examined through the lens of generalizing action and reflection generalization (Ellis, 2007a; 2007b, Table 1). The action generalization describes the learner's mental acts as inferred through the person's activity and talk. Meanwhile, the reflection generalization describes a student's public statement, such as explicitly stating a common property, pattern, or relation of similarity. Categorizing actions and reflections separately also allowed for the identification of iterative *action-reflection-action-reflection* cycles of reasoning, in which students' generalizations evolved in sophistication over time. Each category of action and reflection generalization has some subcategories (Table 1). All of these subcategories are not a process that must be carried out by students. Students may use one of these subcategories in generalizing.

Table 1. The description of action and reflection generalization

Generalization	Categories	Subcategories
Action	Relating	▪ Relating situation: the formation of an association between two or more problems or situation - Connecting back: connecting between a current and previously encountered situation - Creating new: inventing a new situation viewed as similar to both situation ▪ Relating objects: the formation of an association of similarity between two or more present objects - Property: associating objects by focusing on property similar to both objects - Form: associating objects by focusing on their similar form
	Searching	▪ Similar relationships: performing a repeated action to detect a stable relationship between two or more objects ▪ Similar procedures: repeatedly performing a procedure to test whether it remains valid for all cases ▪ Similar patterns: checking whether a detected pattern remains stable across all case ▪ Similar solutions or results: performing repeated action to determine if the outcome of the action is identical every time

Generalization	Categories	Subcategories
	Extending	▪ Expanding the range of applicability: applying a phenomenon to a large range of cases from which it originated ▪ Removing particular: removing some contextual details to develop a global case ▪ Operating: mathematically operating upon an object to generate new cases ▪ Continuing: repeating an existing pattern to generate new cases
Reflection	Identification	▪ Make a general pattern, property, rule, or strategy ▪ Continuing phenomenon, the identification of the property extending beyond a specific instance ▪ Statement of commonality or similarity ▪ General principles: a statement of a general phenomenon
	Definition	▪ Class of objects; the definition of a class of objects all satisfying a given relationship, pattern, or other phenomena
	Influence	▪ Prior idea or strategy: the implementation of a previously developed generalization ▪ Modified idea or strategy: the adaptation of an existing generalization to apply to a new problem or situation

Methods

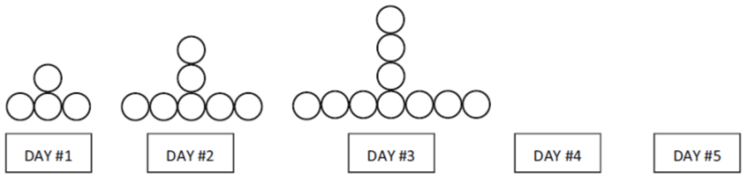
This is a case study that aims to identify elementary students' functional thinking in solving linear pattern-based context problems. Data was acquired from the students' answers and interviews. The case study allows inquiring on selected subjects or phenomena in detail (Cohen et al., 2000). It is an in-depth exploration of a bounded system (e.g., activity, event, process, or individuals) based on extensive data collection. Bounded means that the case is separated out for research in terms of time, place, or some physical boundaries (Creswell, 2012).

The participants in this study were sixty-five grade 5 students (11-12 years old). A test was given to the students. It was a contextual problem about a linear pattern adapted from Wilkie and Clarke (2016). The students with correct answers were grouped into three based on their types of functional thinking and representations (verbal or symbolic). Three students who had different types of functional thinking and representation were purposively selected to be interviewed in a face-to-face and semi-structured format in order to obtain in-depth data about their ways or process of thinking (Fraenkel et al., 2012). An interview protocol used in this study consisted of questions about steps taken by the students in solving the problem. One example of the questions is *how did you get the answer to the number of leaves on the fourth and fifth day? Can you explain the steps?*

The steps of qualitative data analysis: data reduction, data presentation, and verification/conclusion (Miles & Huberman, 1994) were employed. In the data reduction, the students' correct answers were transcribed. The answers were coded based on the types of functional thinking (Confrey & Smith, 1991; Smith, 2008) and representations. The coding scheme for the functional thinking types is recursive patterning (RP), covariational thinking (CT), and correspondence relationship (CR). Meanwhile, the representations consist of verbal representation (RV), table representation (RT), symbolic representation (RS), and other representations (RL). The results of the selected students' interviews were analyzed with the developed rubric referring to actions and reflection generalization (Ellis, 2007a; Table 1).

The "Upside-Down T" Plants test

One day I saw a small plant in the garden with four leaves (day#1). The next day, the plant has grown and has more leaves (day#2). The next day, the leaves grew even more. Every day, I noticed that it continued to grow in the same way.



1. From the pattern above, draw an image like an inverted T crop that will be visible in the next two days (day #4 and day #5).
2. What can you notice about the structure of the plant and the way it grows every day? If you can, color the leaves in the image above a different color to show what you see and explain your thoughts?
3. How many leaves will the plant have on day 7? Explain and show how you got that answer!
4. How many leaves will the plant have on day 17? Explain and show how you got that answer!
5. If someone gives a specific number of days (e.g., P days), how do you find the number of leaves (T) the plant will have in that number of days? Explain and show how you got that answer!
6. On what day does the plant have 100 leaves? Explain and show how you got your answer!

The presentation of the data was performed by describing the students' process of functional thinking for each type obtained from the answers and interviews. In drawing conclusions, the researcher observed patterns of students' answers and compared them with relevant interviews and studies. After obtaining conclusions, verification is carried out by triangulation of theories that are considered appropriate for this research (i.e., Blanton & Kaput, 2004); Lannin, 2005; Stephens et al., 2017; Warren et al., 2006; Wilkie & Clarke, 2016). The verification with theoretical triangulation was held by looking at the similarities of the findings in the current study with the results of previous studies and analyzing the differences in the findings of the research with previous studies.

Findings

The results of the tests show that ten students had correct answers, ten students with incomplete answers, and 45 students with incorrect answers. Of the ten students, two types of functional thinking were identified: *recursive pattern (RP, eight students) and correspondence relationship (CR, two students)*. The students with RP have the same answers and represent the generalization results verbally (RV), while the students with CR represent generalization results in different ways, one student with RV and another student with symbols (RS). Three students were interviewed: S1 (RP and RV), S2 (CR and RV), and S3 (CR and RS).

S1's functional thinking

In solving the problem, S1 thinks functionally by generalizing actions and reflections. In the action generalization, she performed *relating object action* by connecting the information contained in the given problem in the form of leaf shapes (pictures) on day 1, day 2, and day 3, which have the same shape (reversed "T"), but the number is different. She assumed that the leaf was increasing by three leaves per day. The position of the one increased leave is at each edge.

After performing the action, S1 performed the same *relationship searching action*--having repeated actions to detect a stable relationship between two or more objects. She drew leaves on day 4 by drawing leaves on day 3 and then added one leaf at each edge. The same thing was done by S1 to draw leaves on day 5 by using leaves drawn on day 4. In addition, she did the same *pattern searching action* by checking whether the pattern remained stable for all cases. The *relating* and *searching action* of S1 is shown in Figure 2.

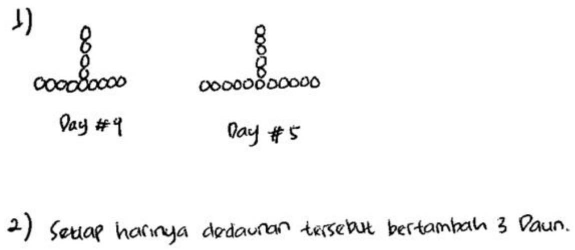


Figure 2. *Relating* and *searching* action by S1

S1 did *extending action* by expanding the range of applicability. That is to apply phenomena to a larger number of cases. In this action, she applied the previously obtained rules to the larger case (7th day and 17th day). For example, to determine the number of leaves on day 7, she started with the number of leaves on day 5 obtained previously and then added 3 for 2 times (day 7= day 5+3+3=16+3 +3=22). Likewise, for day 17, S1 began with day 7 and then added 3 for 10 times. She added the number 3 by 10 because from day 7 to day 17 there are 10 days that must be passed. This *extending action* is shown in Figure 3.

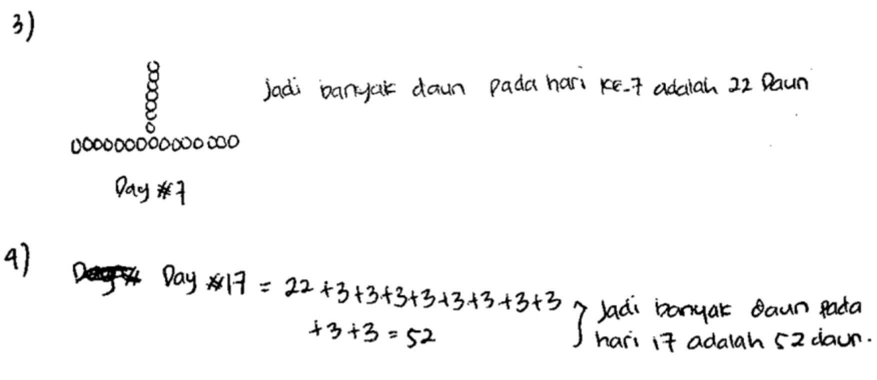


Figure 3. E*xtending* action by S1

After generalizing the action, S1 had a final statement by *generalizing reflection*. She made general principles by identifying general patterns. The reflection is represented verbally, "*every day the leaves increase by three leaves*," and provided with an example "*for example, if you* *determine the number of leaves on day 20, that is to find the number of leaves on day 19 plus 3.*" In determining the inverse, S1 generalized the action by *extending by continuing*, repeating the pattern to obtain new cases. In this case, she continued the pattern found previously in generalizing reflection. She began the calculation from day 17 (*starting*) with fifty-two leaves, then added three leaves repeatedly until the 100 leaves were obtained. Afterward, she calculates how many days are added to get 100 leaves. S1 adds 3 for 16 times or 16 days. Therefore, S1 added up 17 days (*starting*) and 16 days to have 33 days. S1 did not write this process on the answer sheet. This is obtained from the interviews (Transcript 1) with S1 as follows.

Transcript 1

P : *How did you find the number of leaves on the 4th and 5th day?*
S1 : *Firstly, I observed the number of leaves on the 1st, 2nd, and 3rd day and I saw that the leaves were always increasing by 3 every day, increasing by 1 at each edge. So, for the 4th day I drew the 3rd leaf then added 1 leaf at each edge of the leaf and for the 5th day I drew the 4th leaf and added 1 leaf at each edge.*
P : *How did you find the relationship between leaves and days?*
S1 : *As I said earlier, I observed the number of leaves always increasing by 3 each day, increasing by 1 at the three edges.*
P : *What about the 7th and 17th days? Did you do the same?*
S1 : *Yes, I did as in determining the number of leaves on the 4th and 5th day. For the 7th day I drew an inverted T-shaped leaf by looking at the leaves on the 5th day and counting the number of leaves.*
P : *How about day 17?*
S1: *For the 17th day, I did not draw the leaves because there were too many. I found the number of leaves on the 17th day by adding up the 7th day by 3 for 10 times*

From the students' answers and interviews, S1's functional thinking is called recursive-verbal (See Figure 4). This is characterized by: (1) connecting the information obtained from the problems in the form of pictures of leaves on the first and second day, (2) conjecturing a pattern of leaf growth in which the pattern is 3 leaves increasing every day, (3) using the pattern repeatedly for other cases, (4) checking whether the pattern is true for other cases, (5) representing the pattern with words (verbal), (6) using the pattern found to determine the inverse.

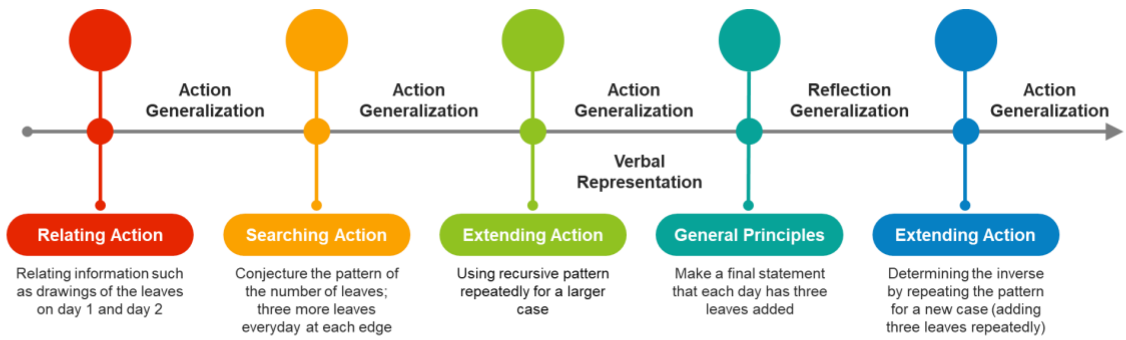


Figure 4. S1's process of functional thinking (*recursive-verbal*)

S2's functional thinking

In solving the problem, S2 thinks functionally by generalizing action and reflection. In generalizing the action, first of all, she carefully understood, grouped, and described the information on the problem. The information obtained by S2 is that the leaves increase in number every day and have a shape like an inverted "T" letter. Other information obtained from the leaf images is about a pattern of the increasing leaf. She realized that the pattern is that every day the leaves increase in multiples of 3 but have a center leaf. At this stage, S2 *relates objects* by connecting two objects (Figure 5) in the form of leaf images on day 1, day 2, and day 3. She assumed that the plant increases by multiples of 3, but the plant has the center leaf.

In the next stage, S2 did *searching the same relationship* action by drawing leaves on the 4th day (Figure 5). To do this, she made one leaf as the center leaf and then added 4 leaves on the top, left, and right sides of the center leaf. In the same way, she drew a leaf on day 5 by drawing one center leaf and then adding five leaves on the top, left, and right sides of the center leaf. Afterward, S2 *searched for some patterns* by checking whether the detected pattern remains applicable for all cases. From the searching action, she conjectured that the number of leaves on the *n*-th day was n times 3 plus 1.

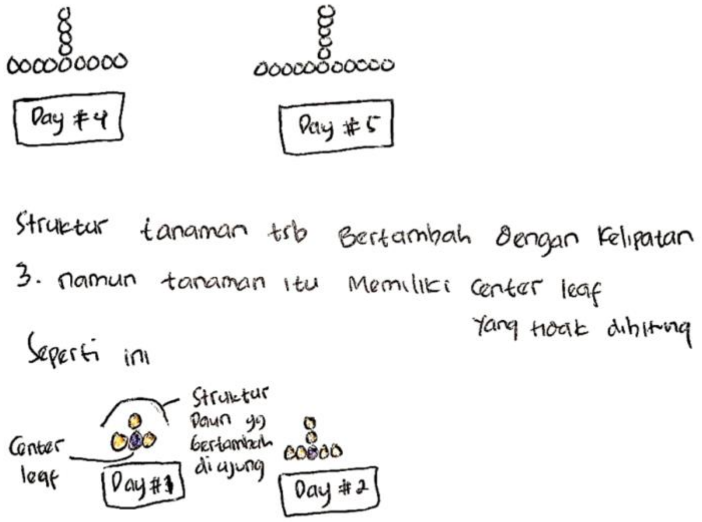


Figure 5. *Relating* and *searching* action by S2

After that, S2 did the *action of extending* by expanding the range of applicability. That is applying phenomena to a larger number of cases. In this action, she used a pattern of multiples of 3 plus the center leaf to determine the number of leaves on another day. For example, to determine the number of leaves on day 7, the student started with one leaf (center leaf) and added 7 multiplied by 3. Similarly, for day 17, S2 started from 1 center leaf and added 17 multiplied by 3.

In generalizing reflection, S2 made *general principles* by identifying general patterns. This reflection was done verbally "*the plant structure is a multiple of 3 but has a center-leaf.*" This verbal generalization of reflection is a form of representation by the students.

In determining the inverse, S2 also *generalized the action* using extending by moving some contexts to develop the general case (*removing particular*). She determined the nearest multiple of 3 from 100 or, in other words, the student subtracted 100 by 1 and then divided by 3 so that students get the number of leaves on day 33. S2 clarified the answers in the interviews (Transcript 2) as follows.

Transcript 2

P : *How did you determine the number of leaves on day 4 and day 5?*

S2 : *First of all, I noticed the number of leaves on day 1, day 2, and day 3 and I observed the structure of the leaves is a multiple of 3 and it has a center. To* *determine the number of leaves on day 4, I drew the center of the leaf first then added 4 leaves at the top, right, and left side. For the leaves on day 5, I did the same thing first and then added 5 leaves at the top, right, and left side.*

P : *How did you determine the number of leaves on day 7 and day 17?*

S2 : *I used the previous pattern, for example, the number of leaves on day 7, I just multiplied 3 then I added the center leaf, I got 22. Likewise, for day 17, I multiplied 3 and added 1.*

P : *How did you relate the leaves and days?*

S2 : *I looked at an inverted T-shaped leaf and noticed that all leaf patterns had a center leaf. I also saw the leaf pattern on day 1 having one leaf on the top, left, and right sides, as well as on day 2 having 2 leaves on the top, left, and right sides. So, I concluded on the specified day, the number of leaves is a multiple of 3 from that day but has a center leaf.*

P : *For the next problem, how did you determine the day when it has 100 leaves?*

S2 : *I firstly subtracted by the center leaf (1 leaf), then the result is divided by 3 and I got 33.*

In general, the process of S2's functional thinking can be summarized in Figure 6.

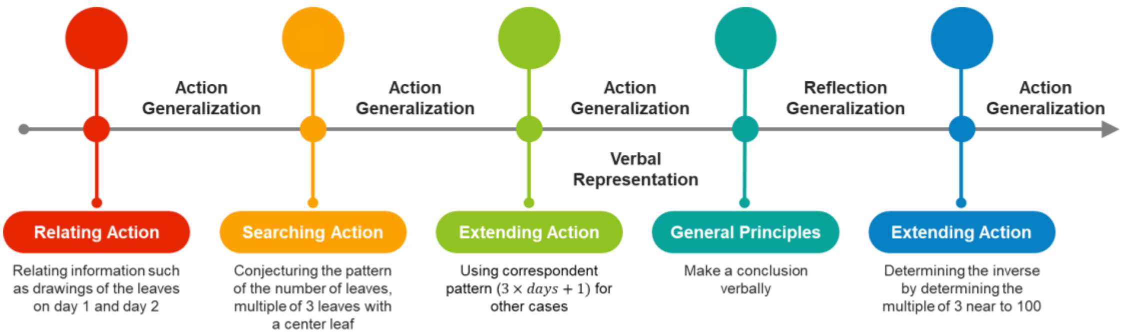


Figure 6. S2's process of functional thinking (*correspondence-verbal*)

S3's functional thinking

In solving the problem, S3 also generalized action and reflection. In generalizing the action, S3 read the problem, analyzed the leaf images, and understood the problem. The reading resulted in the number of leaves that increases every day and having a shape like the inverted "T". The pictures of leaves show the pattern or rules for the number of leaves every day. The student realized that every day the leaves increase by 3; one leaf on each branch (there are 3 branches). In understanding the problem, S3 read each question and then understood in detail what is going to do with it. At this stage, S3 *related objects* by linking information in the form of the leaf drawing on day 1, day 2, and day 3 (Figure 7). She assumed that the leaf is increasing by 3 each day at the edge. After that, S3 *searched for some relationship* by repeating the actions to detect a stable relationship between two or more objects. S3 drew the leaves on day 4 with the following steps:

1. Drew a leaf on the 1st day and gave a sky-blue color to each leaf
2. Added three leaves (1 leaf at each edge) and colored dark blue on each edge to get a leaf image on day 2.

3. Added three more leaves (1 leaf at each edge) at each edge by giving it a pink color to get leaves on day 3.
4. Added three more leaves (1 leaf at each edge) at each edge by giving it a dark blue color to get leaves on day 4

S3 followed the same procedures to draw leaves on day 5 by adding 3 leaves and giving color to each additional leaf. Moreover, she did *the same pattern searching* action by checking whether the pattern remained stable for all cases. After S3 determined the leaves on day 4 and day 5, she did *general principles* by identifying general patterns verbally with the phrase "*every day the leaves increase by 3*." In calculating the number of leaves on day 7 and day 17, she did generalize action again. She related objects by reconnecting the leaves on day 1, day 2, day 3, day 4, and day 5. In this case, she *related objects by properties*. S3 suspected that there was a multiplicative relationship between the number of days and the number of leaves.

Figure 7. *Searching action* by S3

After that, S3 did *the same searching procedure* by repeating the use of the procedure to test valid results for all cases. From *relating* results, S3 estimated that the number of leaves on day 1 is (1×3)+1, day 2 is (2×3)+1, day 3 is (3×3)+1, day 4 is (4×3)+1, and day 5 is (5×3)+1. From the results of relating and searching, she performed an *action of extending* by expanding the range of applicability, namely applying phenomena to a larger number of cases. She determined the number of leaves on day 7 by (7×3)+1=22 and the number of leaves on day 17 by (17×3)+1=52.

After generalizing the action, S3 stated *general principles* by describing the *general formula*. The generalization of the reflection is carried out symbolically by writing the general form of many leaves with the equation "*many leaves on day-p = 3p+1*." In fact, it is a form of representation of the relationship between the number of days and the number of leaves. She represented the relationship of the two quantities symbolically. This representation shows that S3 has a formal *correspondence relationship* thinking.

S3 determined the inverse by moving some contexts to develop a general case (*extending by removing particular objects*). She used the general to determine how many days there are 100 leaves. She made an assumption for the specified day with the variable *h* and then wrote the previous general form for the number of leaves 100, *3h+1=100*. She then subtracted both sides by 1 to get *3h=99*. Finally, S3 divided both sides by 3 to get *h=33*. She clarified the answers in Transcript 3.

Transcript 3

P : How id you determine the number of leaves on day 4 and day 5?

S3 : First, I observed the number of leaves on the first three days and I found a pattern that the number of leaves increased by 3 every day. Therefore, to *determine the number of leaves on day 4, I drew the leaves on day 3 and then added one leaf of a different color at each edge. Likewise, to determine the number of leaves on day 5, I drew the leaves on day 4e, then added leaves of the same color at each edge of the leaf.*

P : *Then how did you evaluate the number of leaves on day 7 and day 11?*

S3 : *I notice the pattern of the number of leaves on the n-th day. For example, on day 1=4 leaves=(1x3)+1, day 2=7 leaves=(2x3)+1, day 3=10 leaves=(3x3)+1, day 4=13 leaves=(4x3)+1, and day 5=16 leaves=(5x3)+1. So, the leaves on day 7 are (7x3)+1=22, the leaves on day 17 are (17x3)+1=52.*

P : *How did you relate the number of days and leaves?*

S3 : *From the calculations I did to find the number of leaves on day 7 and 17, I concluded that the number of leaves on day p is 3p+1*

P : *How did you find the day for 100 leaves?*

S3 : *First, I made an example of the day as **h**, then used the formula 3p+1=100, both sides are subtracted by 1 to have 3p=99 and divided by 3 to get p=33.*

In general, Figure 8 depicts the process of S3's functional thinking.

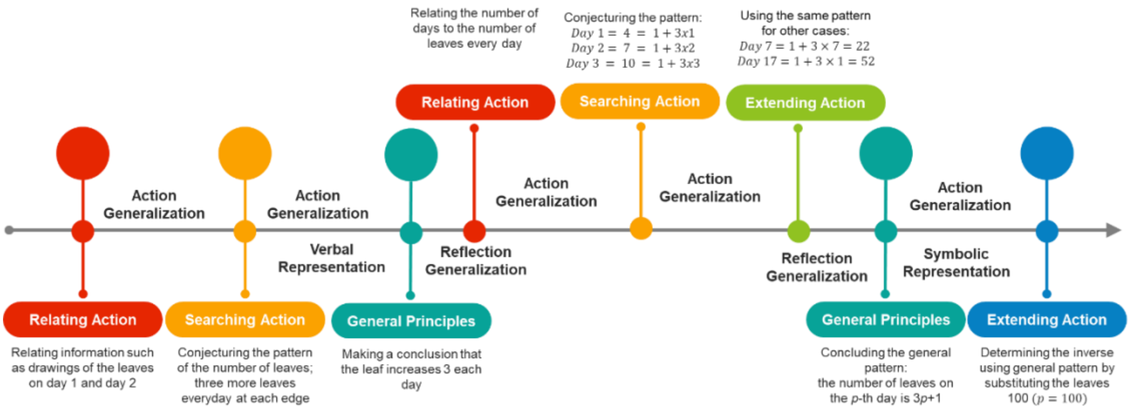


Figure 8. S3's process of functional thinking

The interpretations of the students' answers and interviews reveal their types of functional thinking: *recursive-verbal* (S1), *correspondence-verbal* (S2), and *recursive to correspondence-symbolic* (S3).

Discussion

Several studies unravel that elementary students are able to think functionally (Blanton & Kaput, 2011; Warren et al., 2006; Warren & Cooper, 2005). A similar result was also shown by elementary students in Indonesia in this study. The students are able to think in a functional way, recursively and correspondently. Their functional thinking was analyzed from the generalization of action and reflection. They have the same method in *relating actions* as a part of action generalization, but they have different ways of reflection. In *relating actions*, the students related two or more objects (Ellis, 2007a) (leaf figure at 1, 2, and 3 days), while in reflection, they represent the final statement in different ways.

Based on *searching actions*, there are different strategies that students used in generalizing the relationship between the number of days and leaves. S1 used a recursive strategy (Lannin, 2005; Mouhayar & Jurdak, 2016) by generalizing that the number of leaves increases by 3 every day. Some studies found that elementary students regularly use the recursive strategy in solving linear patterns (Lannin et al., 2006; Mouhayar & Jurdak, 2016; Stephens et al., 2012). S2 used the counting strategy by drawing a picture or constructing a model to represent the situation to count the desired attribute (Lannin, 2005; Mouhayar & Jurdak, 2016). S2 drew pictures of leaves each day, then constructed a model of a linear pattern verbally (leaves increase by a multiple of three but have one center leaf).

Different ways showed by S3 in generalizing patterns. S3 initially used a recursive strategy to determine the number of leaves on day 4 and day 5, then S3 used the contextual strategy (Lannin, 2005) to determine the relationship between the days and leaves. Tanıslı (2011) found that students initially used a recursive approach and looked for a recursive pattern and then found a correspondence relationship in different ways. Ellis (2007a) explains that the reflection generalization categories of the general rule are similar to contextual strategies in the study of Lannin (2003).

Relating and *searching* actions in this study are indicated by determining the number of leaves on day 4 and day 5 while *extending* action is by determining the number of leaves on day 7 and day 17. This follows the near generalization and far generalization by Stacey (1989) where *relating* and *searching* actions are used to determine near generalization, and *extending* action is used to determine far generalization.

The results of this study also reveal students' ability to determine general rules in the generalization of reflection for all cases, both verbally and symbolically (Dindyal, 2007). Students with *recursive-verbal* and *correspondence-verbal* determine the general rules verbally; meanwhile, the students with *recursive to correspondence symbolic* determine the general rules verbally and symbolically. The results of this verbal reflection are similar to intuitive generalization or informal generalization (Amit & Neria, 2008), while symbolic reflection is similar to formal generalization.

In the reflection, the students represent the final statement in a *recursive to correspondence symbolic* way in this study. It is called a constructive generalization, which refers to those direct or closed polynomial formulas that learners construct from the known stages in a figural pattern as a result of cognitively perceiving figures that structurally consist of non-overlapping constituent gestalts or parts (Rivera & Becker, 2011). The students construct general forms (formulas) using figure patterns obtained from the problems. In addition, there is a transfer of representation made by students, from image representation to verbal representation and then to symbolic representation. This transfer of representation is a form of mental flexibility (Amit & Neria, 2008).

In the generalization of action and reflection activities, students go through the experience in decision making, deciding what things stay the same and what changes, what to emphasize and not to emphasize, and what to ignore (Radford, 2006). In this study, the students experienced decision-making to determine the strategy to be used in determining the relationship between the number of days and leaves.

The current study reveals that there are iterative action-reflection cycles of generalization. S1 and S2 go through *action-reflection-action* cycles of generalization. While S3 gets through *action-reflection-action-reflection-action* cycles of generalization. The students did not engage in isolated generalization but produced associated reflection generalization. Ellis (2007a) explicates that students moved between action-reflection generalization, and each influenced the other in a way that allowed the students to develop new knowledge. Also, these cycles are built on previous attempts to develop more sophisticated generalizations.

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Conclusion and Implications

This study found that elementary students can think functionally in solving context-based linear problems in different ways: *recursive-verbal*, *correspondence-verbal*, and *recursive to correspondence-symbolic*. In the *recursive-verbal*, students generalize a relationship between two quantities recursively, represent the generalization results verbally, and determine the inverse by recursive strategy. In the *correspondence-verbal*, students generalize a relationship between two quantities correspondently, represent the generalization result verbally, and determine the inverse correspondently. In the *recursive to correspondence-symbolic*, initially, they generalize a relationship between two quantities recursively, then continue to generalize a relationship correspondently, represent the generalization result symbolically, and determine the inverse using a general rule that has been found.

The results of this study suggest that elementary students in Indonesia have the potential to develop functional thinking, as Afonso and McAuliffe (2019) assert that young learners have the potential to think functionally when offered opportunities to do. Indonesia's mathematics curriculum needs to consider the NCTM principles and standards (NCTM, 2000) for understanding the pattern, relation, and function. Teachers can build generality into their curriculum and instruction (Blanton & Kaput, 2011). Blanton and Kaput (2011) advocate a habit of mind, not just curricular materials, whereby teachers understand both how to transform and extend their current resources so that the mostly arithmetic content of the elementary grades can be extended to opportunities for pattern building, conjecturing, generalizing, and justifying mathematical relationships.

The findings in this study also imply that teachers need to design learning to develop students' functional thinking. Several studies developed learning to see the progress of students' functional thinking (Blanton et al., 2015; Stephens et al., 2017; Warren et al., 2006). Blanton et al. (2015) developed a learning trajectory for enhancing students’ thinking about functional relationships. Moreover, Warren et al. (2006) and Stephens et al., (2017) implemented an instructional sequence to observe the progress of elementary functional thinking.

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