SPECIALIZED FRACTIONS DIVISION KNOWLEDGE: A PROPOSED MODEL

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This paper aims to propose a model of specialized fractions division knowledge (SFDK). The model is drawn from specialized content knowledge (SCK) subdomain (Ball et al., 2008) and a synthesis of prior related works which examine the natures of prospective primary teachers' (PTs) specialized knowledge on fractions division. With respect to preceding relevant studies, the proposed model is more comprehensive since it fully considers translating across representations and different conceptualizations of fractions division. Moreover, it has double functions; to examine and develop PTs' SFDK in the teacher education program.

INTRODUCTION

Having a robust understanding of fractions division (FD) is a major challenge for PTs since, unlike other primary mathematics topics, its characteristics are problematic (Prediger, 2006; Ma, 2010). Researches in the last decades (e.g., Simon, 1993; Li & Kulm, 2008; Olanoff, Lo, & Tobias, 2014) reveal that a multitude of PTs have not fulfilled that challenge. For example, when PTs were asked to create a word problem of the division with fractions, most of them either presented fraction multiplication problem or not able at all to come with the answers (Simon, 1993). Studies, with prior (e.g., Jansen & Hohensee, 2016) and after instruction design (e.g., Adu-Gyamfi et al., 2019) unravel quite similar results regarding the low achievement of PTs' specialized knowledge on the topic.

To help prospective teachers develop such kind of knowledge through a well-prepared design of instructions in the teacher education courses, the information on the nature of their knowledge on fractions division is required (Lo & Luo, 2012; Jansen & Hohensee, 2016). Similarly, that information is vital to examine the effectivity of an instructional design which aims to develop PTs' SFDK. Thus, in order to thoroughly understand such knowledge, a model is definitely needed. This paper attempts to address that need by proposing a model of specialized fractions division knowledge.

PRIOR STUDIES ON SFDK

Olanoff, Lo, and Tobias (2014) extensively reviewed a number of studies which focus on mathematical knowledge for teaching fractions from pre-1998 to 2013.

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Several studies included in the review (e.g., Tirosh & Graeber, 1990; Rizvi, 2004; McAllister & Beaver, 2012; Lo and Luo, 2012) examined the participants' SFDK through tasks which ask them to (1) write words problems from a given number sentences or otherwise, and (2) use pictorial representations to solve word problems. However, the researchers were not concerned about the different conceptualizations of fractions division. For instance, Lo and Luo (2012) examined PTs' specialized fractions division knowledge using a task that asks the subjects to write a word problem which represents 8 2/3 ÷ 1/4 = ? and use drawing to solve it. Three representations are involved in solving the problem.

Two related studies are found after Olanoff's et al. (2014) review. Jansen and Hohensee (2016) examined the nature of PTs' conceptions of a partitive division with fractions prior to the instruction. Referring to the notion of productive conceptions (Llyoid & Wilson, 1998), they set two criteria of conceptions, namely *connected* and *flexible*. Translating between representations and being aware that partitive fractions division generate unit rate are the indicators of *connected conception*. *Flexible conception* is defined as becoming aware that division can involve partitioning, iterating, or both. The results of the study reveal that none of the participants has fully connected conceptions and flexible connection. Within the same objectives to examine PTs' content knowledge and different context, the subjects have participated in the related fractions division course, Adu-Gyamfi et al. (2019) presented three tasks to examine PTs' knowledge regarding conceptualizations and connections the subjects made among diagrammatic, verbal, and algebraic representations. The study did not only cover SCK but also knowledge of content and students (Ball, Thames, & Phelps, 2008) since two items of the tasks present example of students' works to be analysed, whether or not it is a correct solution to the first task.

On the whole, aforementioned studies (e.g., Simon, 1993; Li & Kulm, 2008; McAllister & Beaver, 2012; Lo & Luo, 2012) mostly examined how prospective (elementary or middle school) teachers move from one representation to another, for example from symbolic (number sentences) to words or story problem and different interpretations of fractions division was not its concern. Jansen and Hohensee (2016) focus specifically on one conceptualization of fraction division (partitive) but provide a significant tool to examine specialized content knowledge on partitive fraction division. Adu-Gyamfi et al., (2019) studied the PTs' knowledge which refers to the conceptualizations of fraction division, and connections between verbal, diagrammatic, and algebraic representations but the focus were split to another subdomain of MKT. With respect to the researchers, I argue that the PTs' specialized fraction division knowledge needs to be further comprehensively examined with respect to translating across representations used in solving fractions division problems and different conceptualization of fractions division.

THEORETICAL REVIEW

To strengthen the idea of the model, it is imperative to provide a theoretical review about representation systems of fraction division, different conceptualizations of fractions division, and SCK as follows.

Conceptualization of Fractions Division

The literature agrees that fractions division has diverse conceptualizations (Sinicrope, Mick, & Kolb, 2002; Gregg & Gregg, 2007; Lamon, 2012). Sinicrope et al. (2002) conceptualize fractions division into five; measurement, partitive, unit rate, the inverse of an operator multiplication, and the inverse of a Cartesian product. Some authors (e.g., Gregg & Gregg, 2007; Lamon, 2012) include and use unit rate as part of the partitive division with fractions. The model adopts the five categories but includes only three common conceptualizations; measurement, partitive, and unit rate since they are mostly taught in the classrooms and presented in the textbook (Wahyu & Mahfudy, 2018).

Measurement, partitive, and unit rate interpretations of FD have unique features (Gregg & Gregg, 2007; van de Walle, Karp, & Bay-Williams, 2012; Petit et al., 2016; Jansen & Hohensee, 2016; Shin & Lee, 2018) with respect to *components* (dividend and divisor), *typical situation* (e.g., fair-sharing), *solution process* (iterating or partitioning), and *developed algorithm*. I argue that the ability to *differentiate* each conceptualization is definitely necessary for PTs. For example, when 3/4 ÷ 1/2 is given, two distinct story problems (measurement or unit rate) could be made, or when given two story problems, they could identify which one is for 2 ÷ 3/4 (measurement) and 3/4 ÷ 2 (partitive). After all, this process also associates with translating across representations and thus result in a comprehensive SFDK.

Representations of Fractions Division

External representations are visible productions such as diagrams, graphs, manipulatives, formulas and equations, or mathematical expressions which stand for mathematical ideas or relationships (Goldin, 2014). There are three common external representations widely used and referred to fractions division; *concrete* (e.g., fraction bars), *semi-concrete* (e.g., number lines), and *abstract* such as numerical, verbal or symbolic/algebraic (Adu-Gyamfi et al., 2019). These representations have been deploying to develop students' understanding (Gregg & Gregg, 2007; Wahyu, Amin, & Lukito, 2017) and examine (prospective) teachers' FD knowledge (Lo & Luo, 2012). Solving FD problems involves the translation between verbal representation (word problems), pictorial representations (number lines, area model, or sets of objects model), symbolic representation (number sentences), and algebraic representation which are depicted by the proposed model. In Adu-Gyamfi et al. (2019), algorithm algebraic representation is the algebraic representation of procedures to operate fractions division. In this model, the algebraic representation is the procedures and rationales behind it as well.

One of the key aspects of SCK related to fraction divisions is understanding different representations (Ball et al., 2008). For example, when the PTs is asked to solve a story problem on fractions operations (verbal representations), they are certainly demanded to differentiate which operation fits the story problem, what number sentence (symbolic representation) stands for the problem, what appropriate models (number lines or area model) to construct, what algorithm (algebraic representations) to use and how it relates developed models, and finally all of which lead to correct solution. These processes denote *linking across representations*.

Specialized Content Knowledge

SCK is one of the knowledge components under subject matter knowledge that the teachers should possess. It is defined as mathematical knowledge and skill that is peculiar in teaching (Ball et al., 2008). Referring to the description of SCK presented by Ball et al. (2008), the competencies related to fractions division are (1) differentiating the conceptualization of FD, for example, the difference between measurement and partitive interpretation, (2) linking across various representations in solving FD problems, for instance, write a correct number sentence from a FD story problem, and (3) holding decompressed mathematical knowledge such as explaining why invert and multiply or equalize the denominators to divide fractions. The proposed model is built up by these points. The last point is placed in algebraic representation in the model.

THE PROPOSED MODEL

Drawing from aforementioned components of SCK, prior studies (Jansen & Hohensee, 2016; Adu-Gyamfi et al., 2019), and theoretical reviews on conceptualizations and representations of fractions division, this paper proposes a model of SFDK (Figure 1) which can be utilized to examine PTs' SFDK and prepare an instructional design in a mathematics course to develop such knowledge. I introduce the term *connected and flexible SFDK* adapted from Jansen and Hohensee (2016). *Connected SFDK* is PTs' ability to translate across various representations, not only from verbal to pictorial representations or one direction translation. *Flexible SFDK* is PTs' capability of differentiating measurement, partitive, and unit rate interpretation of fractions division which affect their works on the representations.

The model represents two main components of SFDK, i.e., *linking across representations* and *differentiating conceptualizations of fractions division*. There are two parts of Figure 1. Firstly, diagram inside the large rectangle which denote the first component. Secondly, the rectangle itself that indicates the second component that 'guide' the representations. 'Guide' means that each conceptualization uniquely determines the process of moving from one representation to others. I call it unique since measurement FD has distinct features, e.g., repeated subtraction situation, compared to a fair-sharing situation (partitive FD) which affect the approach students or (prospective) teachers used to solve the FD problems. Two-direction arrow denotes the link of various representations meanwhile the one-direction dashed line denotes the process of solving FD problems. The solution process is included since it entails the way PTs develop decompressed mathematical knowledge as part of SCK.

One example is presented to explicate how the model works. Given this word problem of measurement FD: *Ana is making a flower decoration from 2 1/5 metres ribbon. Each decoration requires 3/5 metre ribbon. How many decorations can Ana make?*

Generally, to solve the problem, PTs could begin by either drawing pictorial representations, for example, number lines or determining number sentence. Let us focus on the first starting point. The problem-context (verbal representation) is translated into number lines (pictorial representation). With the number line, PTs can find the number of decorations (model-based solutions) by partitioning and iterating it. The 3 2/3 decorations are the results of counting how many 3/5s are in 2 1/5 (verbal representation ↔ pictorial representation) **[1]**. However, the process does not stop here since teachers will introduce fraction division to the students.

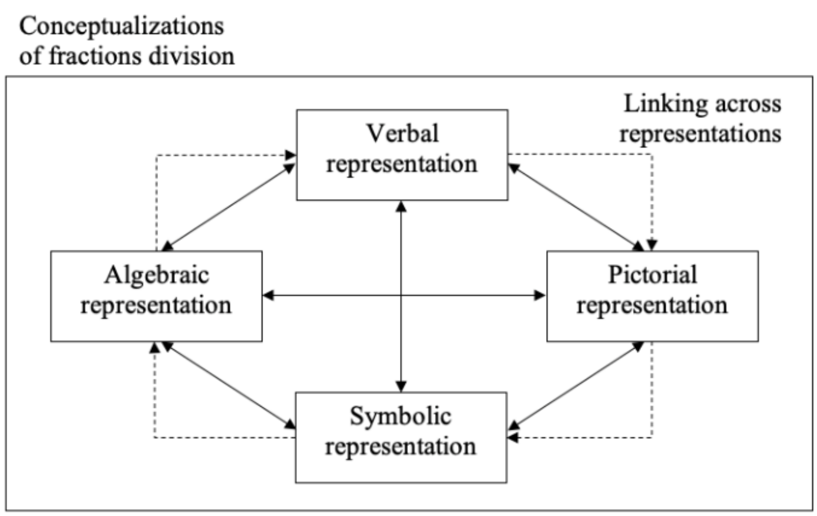

Figure 1: A model of specialized fraction division knowledge

PTs determine number sentence (symbolic representation, 2 1/5÷3/5), or also called as mathematics problem model (verbal representation ↔ symbolic representation) **[2]** which depart from episodic situation comprehension and problem model (Staub & Reusser, 1995). The number sentence is meaningful if

PTs could relate it to the number line. Indeed, the number line represents 2 1/5÷3/5 (pictorial representation ↔ symbolic representation) **[3].** The result of 2 1/5÷3/5 could be determined by using the common-denominator algorithm. In this model, the algebraic representation is not only the algorithm or procedure to calculate the quotient as part of common content knowledge (Ball et al., 2008) but also justification why the procedure could be used to divide. The quotient 3 2/3 and the algorithm are meaningful if PTs could link them to the model-based solutions. 2 1/5÷3/5 = 3 2/3 is similar to determining how many 3/5 metres ribbons are in 2 1/5 metres ribbon. It is the reason why 2 1/5 divided by 3/5 results in 3 2/3. The procedure, 2 1/5 ÷ 3/5 = 11/5 ÷ 3/5 = 11 ÷ 3 = 3 2/3, refers to a number of partitions made in the number line for dividend directly divided by the numerator of divisor or number of iterations based on the divisor. It is the argument why common-denominator algorithm could be used (pictorial representation ↔ symbolic representation ↔ algebraic representation ↔ pictorial representation) **[4]**. At last, the process **[1]** to **[4]** is also linked back to verbal representation and otherwise (linking across representations) **[5]**.

These processes, in my perspective, reflect *connected SFDK*. When PTs are able to translate across representations, from **[1]** to **[5]**, no doubt that they will teach FD conceptually. I also argue that linking across the representations depends on the FD conceptualizations. If the problem is partitive FD, the way PTs do **[3]** is different from measurement since both conceptualizations have distinct situation; repeated subtraction and fair-sharing. Likewise, the unit rate conceptualization is not the same as partitive and measurement when PTs do **[4]** since it uses the invert-multiply algorithm. For this reason, moving across representations is not enough, and that is why differentiating each conceptualization *flexible SFDK* is needed.

Using the model, one could design fraction division tasks to reveal PTs' connected and flexible SFDK. Table 1 shows the exemplary tasks which I am using to test the model empirically. The tasks below can also be utilized to develop PTs' SFDK.

No.	Task
1*	Match the following word problems with the given number sentences! You may use one number sentence for more than one word problem. (1) Dwi has 4 kg of flour to be put in a box. One box contains 2/3 kg of flour, how many boxes does she need? (2) … (7) **Number sentences** (a) $\frac{3}{4} \div \frac{1}{2}$; (b) $\frac{2}{3} \div 4$ (c) $\frac{1}{2} \div \frac{3}{4}$; (d) $1\frac{2}{3} \div \frac{1}{4}$; (e) $4 \div \frac{2}{3}$; (f) $\frac{3}{4} \times \frac{1}{2}$; (g) $\frac{1}{4} \div 1\frac{2}{3}$
2	Use only drawings or models to solve the word problems in number 1! Also, determine the quotient of its number sentence through an algorithm!
3	After solving word problems and determine the quotient in number 2, do you get a similar answer for each pair? If NO, which one is correct? If YES, how your models and algorithm are related? Explain your answer!
4	Write a different word problem for $4 \div \frac{2}{3}$; $\frac{2}{3} \div 4$; $\frac{3}{4} \div \frac{1}{2}$; and $1\frac{2}{3} \div \frac{1}{4}$!
5	What is the difference in words problems for $4 \div \frac{2}{3}$ and $\frac{2}{3} \div 4$? *Hint*: Use contextual problems in number 1. You can compare it to your contextual problems in number 4.

*There are seven word problems in number 1

Table 1: Fraction division task to examine and develop PTs' connected and flexible SFDK

CONCLUSION

This paper explicates the proposed model, which can be used to examine PTs' SFDK and entry point for instructional design to develop such knowledge. It represents a connected and flexible SFDK for which PTs should hold in order to teach fractions division conceptually. Prior studies (e.g., Rizvi, 2004; McAllister & Beaver, 2012; Lo and Luo, 2012; Jansen & Hohensee, 2016) which examine PTs' specialized content knowledge on the topic were limited to one direction of translating between representations and had not fully considered the different conceptualizations of fraction divisions. Nevertheless, SCK (Ball et al., 2008) includes three major components, namely (1) differentiating the conceptualization of FD, (2) linking across various representations, and (3) holding decompressed mathematical knowledge. The proposed model extends the foregoing works and covers all the components.

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