The Potential of Spatial Reasoning in Mediating Mathematical Understanding: The Case of Number Line

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Abstract. Spatial reasoning has been known to have a solid connection to mathematics achievement and the mental mechanism underlying embodied mathematics. However, how spatial reasoning mediates the process of mathematising mathematical ideas is still being investigated. Therefore, this article aims to elaborate on using number lines as a spatial tool to promote spatial reasoning and mathematical understanding. The case is discussed from the perspective of the embodied cognition theory and the instrumentation theory. Based on the theories, the idea of spatialised instrumentation is promoted to explain the nature of spatial reasoning in promoting embodied mathematics learning through spatial learning tools. Under spatialised instrumentation, it is argued that spatial learning tools such as number lines can be used to promote meaningful embodied mathematical experiences involving spatial reasoning that potentially foster the development of mathematical understanding. This finding contributes to the effort of spatialising mathematic learning.

INTRODUCTION

Spatial reasoning is the reasoning that helps in processing space-related information [1]. Spatial reasoning is categorised into three cognitive processes: spatial imagination, spatial interpretation, and spatial representation [2, 3]. Spatial imagination relates to the ability to see or visualise what is said or stated. Spatial interpretation refers to the ability to see or understand what is drawn or seen. Meanwhile, spatial representation is the ability to draw or symbolise what is seen, visualised, or said. The constructs of spatial reasoning can be classified into four categories such as spatial visualisation [4-6], spatial orientation [5, 7, 8], spatial structuring [9-12] and mental rotation [8, 13, 14]. Spatial visualisation refers to the ability to manipulate or transform spatially presented information, such as creating, interpreting, using and reflecting upon pictures, images or diagrams in minds, on paper, or with technological tools to describe or communicate information, thinking or developing ideas, and advancing understandings [4, 6]. Spatial orientation (also referred to as spatial perception [4] or perspective-taking [12]) is the ability to perceive or imagine movement or appearance in space from other locations or perspectives [4, 12]. Meanwhile, spatial structuring is the ability involving the mental process of constructing a spatial organisation of an object or set of objects that reflect the conception of the spatial nature of the object such as recognising its parts, combining the parts into spatial composites, and establishing spatial interrelationships between and among the parts and the composites [9]. Finally, mental rotation is the ability to mentally rotate an object and imagine the spatial information of the object in different positions as the result of the rotation [8].

Spatial reasoning plays a critical role in mathematics education for at least two reasons. First, spatial reasoning has been known to have a solid connection to mathematics competency, especially in arithmetic [5, 15-19]. For example, spatial skills predict kindergarten students' knowledge of numbers and arithmetic proficiency [5]. Spatial skills also significantly determine proficiency in dealing with a calculation involving symbolic numbers [15]. Moreover, spatial

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skills are the predictor of early-grade students' use of higher-level arithmetic strategies [18]. Second, spatial reasoning relates to the notion of embodied cognition, where mathematical ideas are embodied in nature [20-22]. Embodied cognition is a theory that highlights that human knowledge or understanding is influenced and shaped by their physical experiences with their environment [20, 21, 23]. Human knowledge of numbers, for example, is formed through the interactions between human perceptions and their physical experiences of counting, pointing, or measuring [21]. In the embodied instrumentation theory, sensorimotor experiences in a digital environment involve spatial reasoning in promoting mathematical understanding [24]. Moreover, sensory and motoric experiences that are highly spatial in nature influence children's perceptions during the process of constructing mathematical ideas [22, 25-27].

Thus, spatial reasoning is significant in mathematics learning. It implies that mathematics classrooms should promote or encourage students to utilize their spatial reasoning as the means to conceptualize the abstract ideas of mathematics. Several studies have investigated the benefit of integrating spatial reasoning in mathematics learning [see 1, 28]. However, there are many questions that are unanswered yet that are looking for future studies, for example, how spatial reasoning can be used to foster mathematical understanding [29].

Therefore, this article elaborates on how spatial reasoning can be promoted through spatial tools to mediate mathematical understanding. The discussion of this article is focused on the use of a spatial tool namely number line to foster mathematical understanding. The elaboration is divided into three sections. Firstly, the didactical use of number line in mediating students' thinking and reasoning of numbers is presented. Then, the didactical use is discussed from two perspectives, namely the instrumentation theory and the embodied cognition theory. Then, the idea of spatialised instrumentation is promoted to explain the nature of spatial reasoning in promoting embodied mathematics learning through spatial learning tools. It is argued that spatial learning tools, such as number lines, can be used to promote meaningful embodied mathematical experiences involving spatial reasoning that potentially foster the development of mathematical understanding.

THE DIDACTICAL USE OF NUMBER LINE

Number line is an example of a spatial tool that represents numbers horizontally or vertically showing the magnitude of numbers. Number line shows a strong connection between space and numbers. Psychophysical studies, for example, highlight that number line becomes the mental spatial analogy to visualize the magnitude of numbers in space where numbers are spatially represented alongside a horizontal line [30].

Moreover, educational studies show the effectiveness of number line as a concrete and mental tool to contextualize the abstract ideas of numbers into observable objects that allow students to explore the structures and the relationship of numbers [see 31, 32]. For example, the number line is a powerful spatial tool to help students developing meaningful counting strategies and enhance mental computation [31].

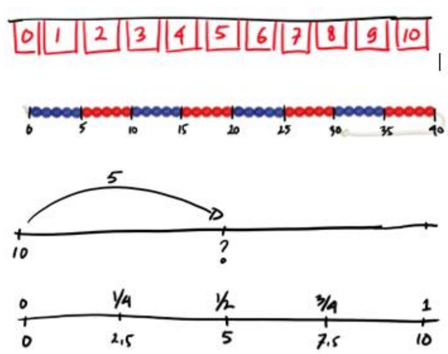

FIGURE 1. Number card string, bead-string number line, empty number line, and double number line.

The use of number line for didactical purposes can be presented in many shapes or forms depending upon the targeted mathematical ideas and students' cognitive level [32]. It can be presented as a string of number cards in the early grades to introduce the sequence of numbers. The bead-string number line can be used to promote the magnitude or the size of numbers. It also can be presented in the form of an empty number line to foster students' number sense, structure, and relationships of numbers. Even fractions can be illustrated through double number lines where one side of the line stands for the unit of reference, and the other side indicates the fractions.

The Magnitude and the Cardinality of Numbers

Students who have lack understanding of the magnitude or the cardinality of numbers may consider 12 and 21 are identical as both numbers are made up from the same numbers, namely 1 and 2. In the context of natural numbers, the magnitude of a number can be defined as the distance of a number from 0. For example, the magnitude of the number 5 is five because the distance of the number from 0 is 5. In this context, the magnitude of a number also shows the cardinality of a number, that a number represents 'how many things'. Here, the number 5 not only defines the distance from 0 (the magnitude) but also represents the number of objects (the cardinality), such as 5 books, 5 oranges, or 5 students. Understanding the magnitude and cardinality of numbers is crucial in early mathematics. It helps students compare the quantity of objects presented in numbers and provides an opportunity for them to explore and understand the relations among numbers which contribute to the development of the number sense. Subitizing is the central idea in understanding both the magnitude and the cardinality of numbers. Subitizing is the idea that a number is constructed from some smaller numbers, but it can be regarded as a whole. For example, six 1s or three 2s can be regarded as a whole 6. It is the foundation for students to recognize the composition of numbers which contributes to students' sense of numbers (recognizing the relations of numbers).

How those big ideas can be promoted to students meaningfully and constructively? Bead-string number line is a powerful spatial representation of numbers that helps students visualize the abstract ideas of both the magnitude and the cardinality of numbers. It can be presented based on 5 or 10 beads depending on the targeted mathematical ideas and students' cognitive level (see Figure 2). Five-based bead-string number line allows students to easily see the multiple or the group of five, such as two 5s is 10 or four 5s is 20. Meanwhile, a ten-based bead-string number line may be more appropriate to explore the concepts of tens and the multiple or the group of tens, for example, two 10s is 20 or three 10s is 30.

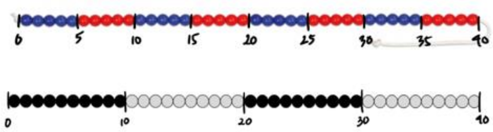

FIGURE 2. Five and ten-based bead-string number line.

In the bead-string number line, the numbers are spatially arranged based on fives or tens starting from smaller numbers on the left and greater numbers on the right. It helps students to compare two numbers by identifying the distance of the numbers from 0 (the magnitude of the numbers) or by identifying the number of beads represented by the numbers (the cardinality of the numbers). For example, in identifying the number of beads represented 12 and 21, by using the five structures, the students may see 12 beads as two groups of 5 beads and 2 beads (2×5 + 2), meanwhile, 21 beads as four groups of 5 beads and 1 bead (4×5 + 1). It is obvious that four groups of 5 beads are more than two groups of 5 beads. Therefore, 12 and 21 must be two different numbers. How 12 is different from 21? The spatial visualization of the five-based number line suggests that 21 is on the right side of 12 as far as 3+5+1 beads, which is 9 beads. Therefore, the difference between 12 and 21 is that 21 is greater than 12 as many as 9 units.

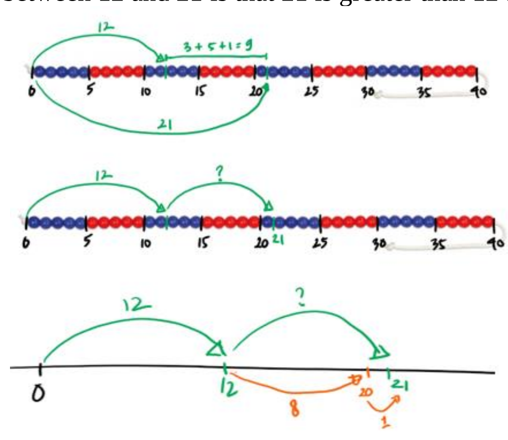

FIGURE 3. Spatial visualization of number line suggests various approaches to see the difference between 12 and 21.

The difference between 12 and 21 can also be articulated in the question, "how many beads that I need to add to 12 beads to get 21 beads?" The spatial visualization of the bead-string number line suggests that we need to add 9 more beads (i.e., 4 blue beads and 5 red beads) to the 12 beads to reach 21 beads (see Figure 3). Moreover, students with a higher understanding can visualize the difference through the empty number line where to go 21 from 12, they may add 8 to 12 to get 20 and then add 1 to 20 to get 21. Here, the spatial visualization of the number line suggests the students to consider the ten-based structure presented in the number line (see Figure 3).

Students can easily show the differences between two numbers by having a mental spatial model of the magnitudes of numbers in the form of lines. For example, in differentiating between 29 and 35, the student can visualize a mental number line in their mind and see that adding 1 to 29 will lead to 30, and then adding 5 to 30 leads to 35. Therefore, the difference between 29 and 35 is 6 where 35 is greater than 29.

The examples above show that the spatial characteristics within the number line, such as the distance from 0 (zero) and the left-right arrangement of numbers, promote students to construct a spatial analogy of the magnitudes and the cardinality of numbers and use the analogy to explore and describe the relations among numbers. The spatial analogy refers to the use of spatial relationships (e.g., line) to explain the relationships of non-spatial ideas (e.g., the magnitude of numbers). Here, the spatial analogy scaffolds students to think, explore, and describe the relationships of the abstract ideas of the magnitude and the cardinality of numbers.

The Magnitude and the Cardinality of Numbers

The spatial analogy of numbers in the form of a number line provides scaffolding for students to think, develop and communicate their own counting strategies. Understanding the magnitude of numbers, the problem, such as 34 + 26, can be visualized on the mental number line as adding 26 beads to 34 beads (see Figure 4). The spatial visualization triggers students to think of numbers between 34 and the question mark, which help them determine the question mark's magnitude. This suggests students split 26 into some parts that would be easier to add to 34, for example, splitting 26 into 10+10+6, 6+10+10, 20+6, or 6+20. The choice of the splitting is determined by the characteristics of the numbers being operated. For example, it would be easier to split 20 into 6+20, rather than 7+19, since it is simpler to add 34+6+20 (i.e., 40+20) rather than 34+7+19 (i.e.,41+19) due to the underlying ten-based structure. The various ways of splitting 26 allow students to imagine various strategies to solve 34+26.

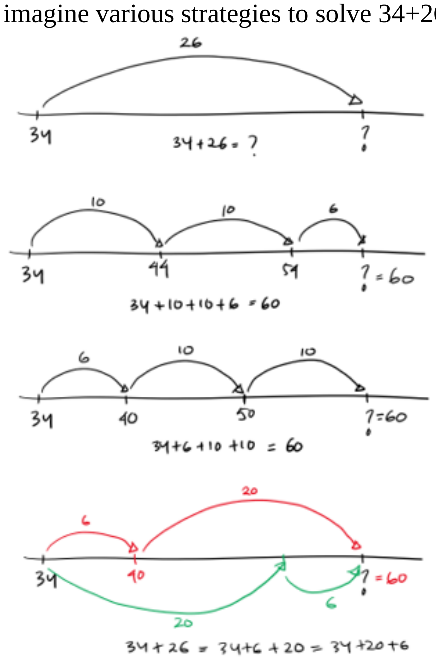

FIGURE 4. Decomposition strategies to solve 34+26 on the number line.

The experiences of splitting will lead students to invent the decomposition method in dealing with addition. The decomposition method is a counting strategy in that numbers are split or decomposed into their compositions such that it would be easier to recompose. For example, through the number line visualizations, students could invent mental imagination that 34+26 can be seen as 34+6+20 (split and counting on) or as 30+20+4+6 (split tens and ones) [33].

In the context of subtraction, the spatial analogy of numbers on number line scaffolds students to invent and understand the constant difference principle [31] that other subtractions can be generated from a subtraction by omitting or adding the same amount of number in both minuend and subtrahend to preserve the difference. For example, 67 – 43 equals 47 – 23 since 20 is taken from both 67 and 43 resulting in 47 and 23, respectively. Understanding the constant difference principle helps simplify subtractions involving complex numbers. For instance, 397 – 179 can be transformed into other equal subtractions by gradually omitting the same number on both the minuend and the subtrahend. For example, omitting 100 from 397 – 179 leads to 297 – 79, then omitting 7 leads to 290 – 72, then omitting 70 leads to 220 – 2. Therefore, solving 397 – 179 can be done by solving easier subtractions, such as 220 – 2. Moreover, adding and omitting numbers could provide a powerful shortcut to solve subtraction. For example, 359 – 297 can be viewed as 162 – 100 by adding 3 and then omitting 200.

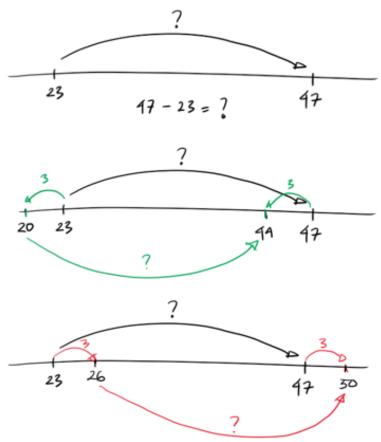

FIGURE 5. Constant difference strategies to solve 47 – 23 on the number line.

The power of the constant difference principle can be promoted to students constructively and meaningfully by exploring the number line. Consider the subtraction 47 – 23. Understanding the magnitude of numbers, 47 – 23 can be visually defined on the number line as the distance between 23 and 47 (see Figure 5). The student can then be asked to shift one number (either minuend or the subtrahend) to their preferences and think of how the other number should be shifted to preserve the difference. Suppose 23 is shifted backward to 20 (omitting 3 from 23). To keep the difference, 47 should be shifted backward as far as 3 reaching 44. Therefore, 47 – 23 can be seen as 44 – 20 which is easier to be solved. Moreover, 47 – 23 can be viewed as 50 – 26 by shifting the numbers forward as far as 3 units. If the difference is visualized as a bar, the shifting can be regarded as moving the bar forward or backward to be adjusted with the familiar numbers (see Figure 6).

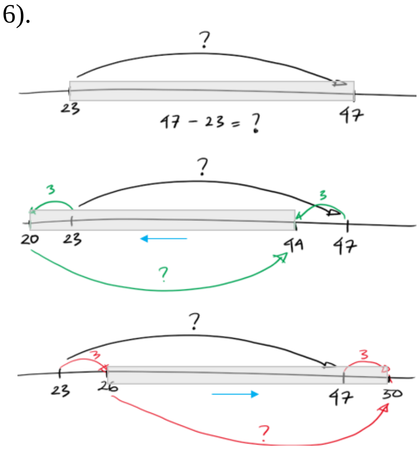

FIGURE 6. Constant difference strategies are illustrated as moving a mental bar to familiar numbers.

The examples illustrated above show how number lines could interact with students during knowledge construction about numbers. The properties of numbers line, such as their spatial representations, supply perceptions to students' minds where such perceptions trigger students to act (e.g., drawing arrows on the number line when adding or identifying the familiar closest numbers). The continuous interplay between the spatial perceptions supplied by the number line and the related actions contribute to knowledge constructions. The term "spatialized instrumentations" is proposed to explain such phenomena. It is based on the embodied cognition theory and the instrumentation theory, which are elaborated on in the following sections.

NUMBER LINE, SPATIAL REASONING, AND EMBODIED COGNITION

Generally, spatial reasoning defines our ability to deal with space. This reasoning can be categorised into three types, namely spatial imagination, spatial interpretation, and spatial representation [2, 3]. Spatial imagination relates to the ability to visualise what is said or stated verbally. Spatial interpretation refers to the ability to understand what is drawn or seen. Meanwhile, spatial representation is the ability to draw what is seen, visualised, or said.

Spatial reasoning is naturally embodied [22]. It implies that spatial reasoning involves a bodily engagement in dealing with space where the potentials of our body, such as sensory and motoric skills, are utilised in making sense of our spatial world. Therefore, spatial reasoning is strongly related to the concept of embodied cognition, a concept highlighting the strong connection between the human body, mind, and environment in knowledge construction (see Figure 7).

The basic claim of embodied cognition is that our knowledge or mind is shaped by the experience of our body within the environment [25, 34]. According to Piaget, knowledge is constructed through the process of assimilation and accommodation [35]. Assimilation refers to the process of adjusting new information within the existing knowledge. Meanwhile, accommodation is the process of replacing the pre-existing knowledge as new relevant knowledge is accommodated. One of the prominent causes of assimilation and accommodation is our physical interactions with our environment. The interactions involve our sensory and motoric experiences (sensorimotor experiences), which are facilitated by the affordances of our body and environment.

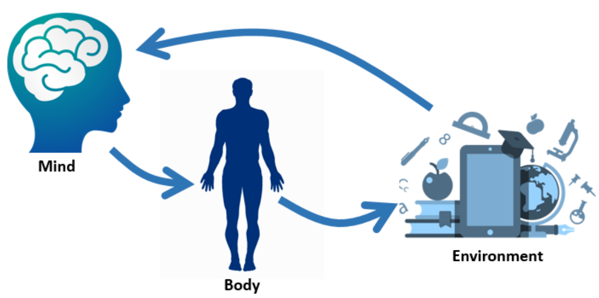

FIGURE 7. Embodied cognition: the interactions between mind, body, and environment in knowledge construction.

In the context of mathematics, although they are highly abstract, mathematical concepts are acknowledged to be rooted in sensorimotor experiences [20, 21, 24, 36]. Therefore, it is embodied in nature. It can be seen from the facts that human understanding of numbers is constructed through the interactions between the human mind and their physical experience of counting, measuring, and pointing [21]. In addition, our knowledge of the mathematical ideas in geometry is shaped by our physical interactions with geometrical objects in our environment [37]. This evidence highlights that our mathematical understanding is not only the product of our mind but also influenced by our physical interactions with our environment [20, 23]. Here, our knowledge and understanding on some level are conditioned by our body's capabilities and constraints in interacting with our environment [36].

How is the number line related to embodied cognition? As spatial reasoning is embodied in nature [22], spatial reasoning could be used as the mediator to explain the embodied nature of the number line. Once students are working with a number line, the potential of their sensory and motoric skills are stimulated due to the visual and motoric properties of the number line. The visual properties of a number line refer to the visual linear representation of numbers on a line. The representation could reveal and trigger students to think about the relation of various mathematical ideas simultaneously, such as the sequence, the magnitude and the cardinality of numbers. Meanwhile, the motoric properties of the number line refer to the affordances of the number line to involve physical actions, such as drawing an arrow or pointing a point on the line, to represent or express mathematical thinking or reasoning (see Figure 4).

Here, the sensory and motoric properties of the number line trigger the potentials of our physical body (i.e., sensory and motoric skills) to gain sensory and motoric experiences produced by the number line, which contribute to the creation of perceptions in our mind about the experiences (see Figure 8). The created perceptions then shape or influence how we react (actions) towards or work with the number line. These actions are classified into types. The first is the actions that our mind gives to our body which are unobservable (mental actions). The second is the actions of our body toward the tools, which are relatively observable (observable actions).

The reciprocal relationship between the perceptions and actions involving the potential of our body forms perception-action loops (see the dark blue arrows in Figure 8). Here, the sensory and motoric skills of our body, which naturally involve spatial reasoning together with the number line supply perceptions in our mind, and then based on the perceptions, our mind gives reaction resulting in the perception-action loops which contribute to knowledge construction [25]. Once the perceptions of the properties of an object, such as number line, are created or crystallised in our mind, we could use the perceptions to imagine the object and mentally react on the object without physically engaging with the object. This condition creates mental perception-action loops (see the light blue arrows in Figure 8).

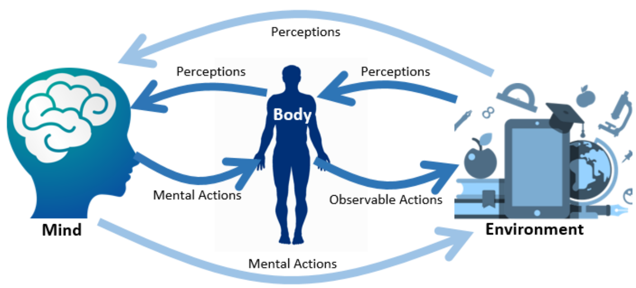

FIGURE 8. The role of the body in knowledge construction.

Thus, regarding perception-action loops created by working on a number line, knowledge construction of numbers can be effectively facilitated through number line exploration. It provides students with numerous embodied experiences involving spatial reasoning, sensory, and motoric experiences that help them experience mathematical ideas meaningfully. Thom, D'Amour [22] call such a spatial experience embodied mathematics. In this article, the term "spatialized instrumentation" is introduced to express the idea that is relevant to embodied mathematics by focusing on the role of spatial reasoning generated from the embodied experiences with spatial tools in knowledge construction. The following section elaborates on the notion of spatialized instrumentation about the number line.

NUMBER LINE AND SPATIALISED INSTRUMENTATION

Regarding the critical role of spatial reasoning in knowledge constructions under the embodied cognition theory [22], the notion of spatialized instrumentation is proposed referring to the idea that knowledge construction can be facilitated through designed spatial tools or experiences that produce spatial reasoning, sensory, and motoric experiences. Besides the embodied cognition theory, the spatialized instrumentation is based on the notion of the instrumentation theory, a concept explaining the interactions between learning tools and users during knowledge construction.

The central idea of the instrumentation theory is the argument that the affordance of a learning tool influences the construction of the knowledge of its users. At the same time, the users' pre-existing knowledge affects how the tool is utilized as an instrument [24-27]. The reciprocal relationship between tools and users is defined by two mechanisms called instrumentalization and instrumentation (see Figure 9). The instrumentalization defines the mechanism by which users' pre-existing knowledge influences the way a learning tool is used. Meanwhile, instrumentation refers to the idea that the affordances of learning tools shape users' knowledge. During the instrumentalization, the influence of users' pre-existing knowledge toward the learning tool is operationalized through actions. For example, students draw an arrow to express a jump when doing addition on number line. Meanwhile, during the instrumentation, the properties of the learning tool supply perceptions about the tool. For example, as students look at the locations of tow numbers on a number line, they realize that there may be many other numbers between the two numbers, such as number 20 or 21 between 18 and 24 (see Figure 10). The perception and action form a reciprocal relationship where learning tools supply perceptions to our mind, and our mind reacts based on the perceptions. The reciprocal relationship between perceptions and actions can be described as perception-action loops that contribute to knowledge construction [25].

Furthermore, the human body could also be regarded as a learning tool on its own as our body is not part of our mind, and our mind could give actions (instructions) to our body to do our intended actions. Therefore, our bodily experiences with a learning tool supply perceptions to our mind and, at the same time, the created perceptions promote reactions toward our body and the learning tool. Here, the body and the tool are regarded as learning tools that supply perceptions to our minds. The interplay between perceptions and actions involving our body in the process of knowledge construction is called embodied instrumentation [24, 25].

Knowledge construction occurs as the result of the interplay between actions and perceptions during instrumentalization and instrumentation. Regarding the interplay, instrumentation theory suggests the importance of selecting or designing appropriate learning tools to target the construction of the intended knowledge or understanding. For example, if we intend to build students' understanding of subtraction as the difference between minuend and subtrahend, learning tools that could effectively promote such understanding is necessary to be designed or selected.

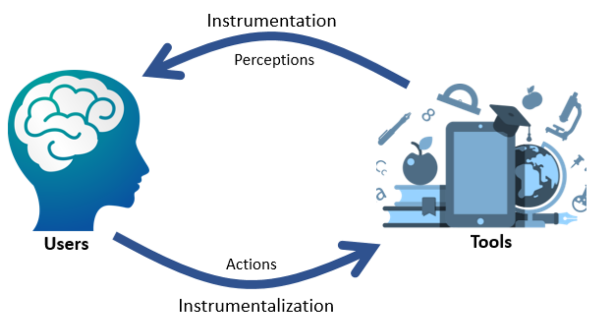

FIGURE 9. perception-action loops in the instrumentation and the instrumentalization.

How could number line construct students' knowledge according to the instrumentation theory? Given the examples of the didactical use of number line in the previous section, it is obvious that number line plays a critical role as a learning tool that mediates students' knowledge construction of numbers. The representation of number line triggers and simultaneously allows students to express their thinking and reasoning. It helps students to make sense of their thinking and reasoning. Here, the number line transforms the abstract meaning of mathematical ideas (e.g., the concepts of numbers) into imaginable and visible ideas (e.g., numbers as line segments or distance), therefore, it is relatively accessible for students to think and reason with. Furthermore, a number line could become a mental model or a mental analogy of numbers that allows us to think about numbers without physically engaging with concrete representations.

Number lines could shape the construction of students' knowledge of numbers because of their spatial properties, namely their visual and motoric properties. The visual properties refer to the visual representation of numbers on a number line where the way the numbers are presented reveals various mathematical ideas simultaneously, such as the sequence of numbers, the magnitude, the cardinality, and their relationships. Meanwhile, the motoric properties refer to the affordances of the number line to involve physical actions, such as drawing an arrow, to represent or express mathematical reasoning or ideas.

How the two properties contribute to the process of instrumentalization and instrumentation? To answer this question, consider the following problem, 24 – 18 = ?. The number line presentation of the problem creates the perception that 24 – 18 is the distance between 18 and 24 (see Figure 10). The perception, then, contributes to several mathematical actions to find the distance. One possible action could be adding a line segment on18 to make 18 equal 24. The length of the added line segment would be 6 because 18+6 is 24. Therefore, 24 – 18 is 6. The other possible action could be removing a part of the line segment and constructing 24 to make 24 equal 18. The length of the removed line segment is the solution for 24 – 18 which is 6. As the students try to actualize the action of adding and removing, they could draw arrows to show or track their thinking. This example shows that the experiences of working on the number line could construct students' knowledge of subtraction, such as the subtraction of two numbers is the distance between the two numbers and the distance could be a number that makes minuend and subtrahend equal if it is added on the subtrahend or removed from the minuend. The knowledge can be presented mathematically as p – q = s if q + s = p or p – s = p.

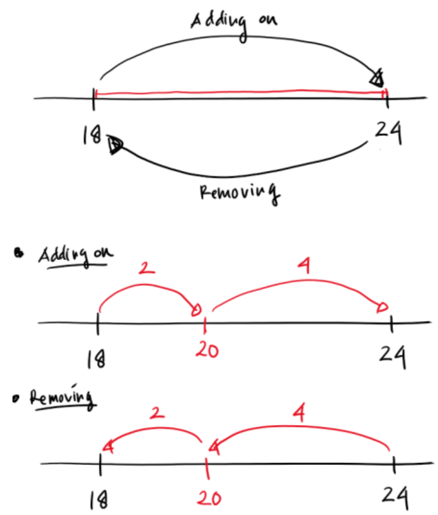

FIGURE 10. The spatial representations of defining and solving 24 – 18 on a number line.

The example above shows us that the properties or the affordances of a spatial tool, such as a number line, supply spatial perceptions to our minds about the tool (see Figure 11). The perceptions are acquired by the potential of our body to access the spatial properties of the tool through our sensory and motoric skills (see the dark blue arrows in Figure 11). The acquired spatial perceptions are then processed in our mind to be related, assimilated or accommodated with the relevant pre-existing knowledge. This process generates our knowledge about the ideas underlying or expressed through the tool. This knowledge then is used to give responses in the forms of actions where the actions are executed by the aids of our body's sensory and motoric skills. Here, the sensory and motoric potentials of our body mediate the interplay between our mind and the tool through the perception-action loops during instrumentalization and instrumentation. As students continuously work with a spatial tool, it creates multiple related spatial perception-action loops that contribute to the knowledge construction.

Once the perceptions about the spatial tool have been crystalized in our minds, we could still gain perceptions from the tool without physically engaging with the tool (see the light blue arrows in Figure 11). Here, the tool becomes our mental model that helps us work with the tool in our mind to explore further the underlying mathematical ideas represented through the tool. This response is called a mental action. The crystalized perceptions about the tool and the mental actions generate mental spatial perception-action loops. Such loops could contribute to the construction of a spatial analogy of non-spatial ideas. For example, we could create a mental number line as an analogy to think about the relations of the magnitude of numbers or to solve computations. For instance, our mental number line suggests that 17+9 can be solved by taking 3 from 9 to be added to 17, resulting in 20 and adding the remaining 6 to 20, resulting in 26. Alternatively, since 9 is close to 10 on the number line, 17+9 can be seen as taking 1 from 17 to be added to 9 resulting in 10 and then adding 10 and the remaining 16 which is 26.

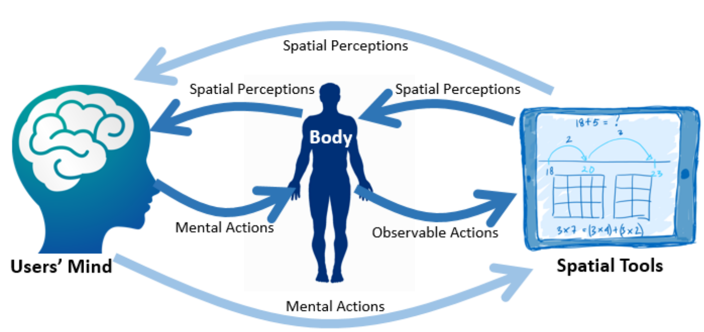

FIGURE 11. The mechanism underlying the spatialized instrumentation during knowledge construction.

Each spatial tool has its own spatial properties that could generate different spatial experiences and, consequently different spatial perception-action loops. In the context of multiplication, for example, the number line and array model expose different spatial experiences (see Figure 12). For instance, the number line representation of 3×4 suggests the multiplications as a distance created by adding 4 as many as 3 times. Meanwhile, array representation suggests 3×4 as an area having 3 rows of units where each row has 4 units. Moreover, although both representations could trigger students to see multiplication as repeated addition, the array model is more suggestive in inciting students to see multiplications as equal grouping (e.g., 3 groups of 4) and the commutative principle of multiplication (e.g., 3×4 = 4×3). As the spatial properties of the number line and array model are different, they might generate other spatial perception-action loops that contribute to different processes or forms of knowledge construction. This implies that spatial learning tools must be designed or selected purposively to meet the intended learning goals.

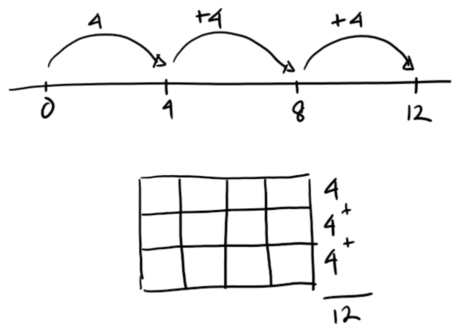

FIGURE 12. Number line and array: Different spatial representations for 3×4.

CONCLUSION

This study can be considered an alternative approach to promoting spatial reasoning in mathematics learning. It is stimulated by the question of how spatial reasoning can be used to facilitate mathematics learning. Based on the embodied cognition and the instrumentation theory, the idea of spatialized instrumentation is introduced to explain the critical role of spatial reasoning in knowledge construction. It is conjectured that developing mathematical knowledge can be facilitated through designed spatial tools together with the attached learning tasks. The case of number line suggests that students could construct their mathematical understanding by exploring the properties of numbers through the spatial representation.

The spatialized instrumentation views mind, body, and tool as an integrative system in knowledge construction through the reciprocal relationships of perception and action (perception-action loops). This idea might be related to the idea of the embodied mathematics [22] or the embodied instrumentation [25]. Both ideas suggest the importance of bodily experiences in knowledge constructions. However, the spatialized instrumentation highlights the critical role of spatial experiences in knowledge constructions where the experiences might be occurred bodily (physical engagement with tangible spatial learning tools) or mentally in our mind without physical engagement of our body (the mental spatial perception-action loops). The mental spatial experience is generated from the embodied spatial experience through concrete or tangible spatial tools, such as drawing number line.

It is believed that each spatial experience may lead to a different form of spatial perception-action loops resulting different form of knowledge construction. For example, the number line and the array representation of multiplication could produce dissimilar spatial perceptions of multiplication and provoke slightly different conceptions of multiplication. Other than through spatial representations, spatial experiences can be exposed to students through other methods that generate spatial experiences, such as body-based tasks where students are asked to perform spatial tasks with their body based on the given instructions. Therefore, future research needs to investigate the spatial perception-action loops of the various types of spatial experiences. For example, the study aims to investigate students' knowledge as the result of using two different spatial experiences.

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